STABILITY OF TRANSPORT PROCESSES IN CONTINUOUS MEDIA WITH HEAT OR MATERIAL SOURCES AND SINKS

P. P. Lazarev and A. S. Pleshanov UDC 536.24.02

A sufficient stability criterion is obtained for the heat-conduction process in a continuous medium with variable coefficient of heat conduction and heat source.

Although this investigation is equally valid for both heat propagation because of heat conduction and for material transport because of diffusion, for definiteness we shall nevertheless speak everywhere here about heat propagation.

The stability of transport processes in continuous media with heat sources has been investigated in a number of papers [1-5]. A variational treatment of this question, different from [1-5], is elucidated here.

The stationary energy equation has the form

$$\operatorname{div}(\varkappa\nabla T) + Q = 0, \tag{1}$$

where T is the temperature, $\varkappa = \varkappa(T)$ is the heat-conduction coefficient, $Q = Q(T)$ is the volume intensity of evolution (heat absorption).

The functional J whose Euler equation is (1) is

$$J = \int \left[\frac{1}{2}\ \varkappa^2(\nabla T)^2 - \int\limits_{T_0}^{T} \varkappa Q dT\right] dV, \tag{2}$$

where T_0 is some reference point of T and integration is over the volume V. In fact, the first variation of $J(\delta J)$ after the utilization of Green's theorem has the form

$$\delta J = \int \varkappa^2 \delta T (\nabla T dS) - \int \varkappa [\operatorname{div}(\varkappa\nabla T) + Q]\, \delta T dV, \tag{3}$$

where the integral over the surface S is zero for fixed values of T and S, and the condition $\delta J = 0$ results in (1) for arbitrary δT within the domain.

The nonstationary equation corresponding to (1) is

$$\rho c\ \frac{\partial T}{\partial t} = \operatorname{div}(\varkappa\nabla T) + Q, \tag{4}$$

where ρ is the density and c is the specific heat.

Considering T in (2) to satisfy (4) and, therefore, J to depend already on t, we find the rate of change of J. Using (4), we obtain analogously to (3)

$$\frac{\partial J}{\partial t} = \int \varkappa^2\ \frac{\partial T}{\partial t}\ (\nabla T dS) - \int \rho c \varkappa \left(\frac{\partial T}{\partial t}\right)^2 dV. \tag{5}$$

For $\delta T = 0$ the quantity $\partial J/\partial t$ is negative on S because of $\rho c \varkappa > 0$. The stationary (extremal) value $J = J_0$ is achieved upon compliance with the condition (1); i.e., $\delta J_0 = 0$. However, for the value of J_0 to be stable it is necessary that this extremum of J be a minimum for $\partial J/\partial t < 0$; i.e., $\delta^2 J_0 > 0$.

It is easy to see that

$$\delta^2 J_0 = \frac{1}{2} \int \frac{\partial \varkappa^2}{\partial T}\ (\delta T)^2 (\nabla T dS) + \frac{1}{2} \int \varkappa^2 \left\{(\nabla \delta T)^2 - \left[\frac{\partial^2 \ln \varkappa}{\partial T^2}\ (\nabla T)^2 + \frac{\partial Q/\varkappa}{\partial T}\right] (\delta T)^2\right\} dV. \tag{6}$$

Therefore, for $\delta T = 0$ on S, compliance with the condition

G. M. Krzhizhanovskii Energy Institute, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 35, No. 2, pp. 334-337, August, 1978. Original article submitted June 6, 1977.

$$\frac{\partial^2 \ln \varkappa}{\partial T^2}(\nabla T)^2+\frac{\partial Q/\varkappa}{\partial T}<0 \tag{7}$$

is sufficient for $\delta^2 J_0 > 0$. In particular, for $\varkappa$ = const condition (7) is satisfied for $\partial Q/\partial T < 0$, which holds in the case of Joulean dissipation, e.g., when $Q = j^2/\sigma$ (j is the electrical current density, assumed constant, and σ is the electrical conductivity which grows with T).

The reverse inequality to (7) is a necessary (but not sufficient) condition for instability. In the case of Joulean dissipation when the dependence of σ on T is quite strong, necessary for compliance with the last criterion is that $(\partial^2 \ln\varkappa/\partial T^2)_p > 0$ and, moreover,

$$\left(\frac{\nabla T}{j}\right)^2>-\left(\frac{\partial(\varkappa\sigma)^{-1}}{\partial T}\right)_p\left(\frac{\partial^2 \ln \varkappa}{\partial T^2}\right)_p^{-1},$$

where p is the pressure. The condition $(\partial^2 \ln\varkappa/\partial T^2)_p > 0$ is satisfied on sections abutting the minimums of the equilibrium curve of $\varkappa$ as a function of T for the parameter p. These sections exist during the passage of equilibrium chemical reactions in a medium.

In the case of nonequilibrium chemical reactions, when Q can be treated as a heat source (or sink), the criterion (7) cannot be satisfied for $\varkappa$ = const. This holds for exothermal reactions when $Q \sim \exp(-E/T)$, where $E > 0$ is the activation energy.

Here the stability investigation should be conducted by solving a variational equation which is obtained upon varying (1). This equation, called the Jacobi equation, is simultaneously the Euler equation of functional (6) (see [6], for instance).

For a perturbed regime deviating slightly from (1), Eq. (4) has the linear form

$$\rho c\,\frac{\partial T'}{\partial t}=\Delta(\varkappa T')+\frac{\partial Q}{\partial T}\,T'. \tag{8}$$

Considering the perturbation $T' \to T'(x)\exp(kr)$, where r is the radius in the y – z plane and k is the appropriate wave vector, we obtain from (8)

$$\rho c\,\frac{\partial T'}{\partial t}=\frac{\partial^2(\varkappa T')}{\partial x^2}-\left(k^2\varkappa-\frac{\partial Q}{\partial T}\right)T'. \tag{9}$$

Here all the coefficients are functions of the unperturbed value of T and, therefore, are functions of x. The functional in T' whose Euler equation is the right-hand side of (9) has the form

$$J'=\int\left[\frac{1}{2}\left(\frac{\partial \varkappa T'}{\partial x}\right)^2-\varkappa\left(k^2\varkappa-\frac{\partial Q}{\partial T}\right)\frac{T'^2}{2}\right]dx.$$

Its second variation is

$$\delta^2 J_0'=\frac{1}{2}\,\varkappa\,\frac{\partial \varkappa}{\partial x}(\delta T')^2+\frac{1}{2}\int \varkappa^2\left\{\left(\frac{\partial \delta T'}{\partial x}\right)^2+\left[k^2-\frac{\partial^2 \ln \varkappa}{\partial T^2}\left(\frac{\partial T}{\partial x}\right)^2-\frac{\partial Q/\varkappa}{\partial T}\right](\delta T')^2\right\}dx,$$

and the sufficient stability condition

$$\frac{\partial^2 \ln \varkappa}{\partial T^2}\left(\frac{\partial T}{\partial x}\right)^2+\frac{\partial Q/\varkappa}{\partial T}<k^2 \tag{10}$$

is weaker than the analogous condition (7).

Therefore, taking account of periodic perturbations in a plane perpendicular to the direction of the unperturbed distribution of T results in greater stabilization of this distribution. In other words, longitudinal perturbations (in the distribution) are most dangerous: If the system is stable with respect to them, then it will be all the more stable in the presence of transverse perturbations. On the other hand, if the system is unstable with respect to the longitudinal perturbations, then switching in sufficiently shortwave transverse perturbations will stabilize the system.

As mentioned above, the Jacobi equation must be solved in the case of noncompliance with the sufficient stability conditions (7) or (10). If $\varkappa$ = const and $Q \approx \exp(\alpha T)$, then analytic solutions of both the Euler and Jacobi equations and of the stability problem in its traditional formulation can be obtained in the one-dimensional case. Th following fact [7] is the foundation for this.

The expression

$$y = \varphi \left[\frac{1}{\varphi} \left(C_+ e^{\lambda x} + C_- e^{-\lambda x} \right)' \right]'$$

is a solution of the equation

$$y'' = [\lambda^2 - 2(\varphi'/\varphi)']\, y$$

under the condition

$$(\varphi'/\varphi)'' + 2(\varphi'/\varphi)(\varphi'/\varphi)' = 0.$$

Here the primes denote differentiation with respect to x, λ = const is a parameter, $C_\pm$ are constants of integration, and $\varphi = \sin x$, $\cos x$. On making the replacements x, $\lambda \to$ ix, iλ, we have $\varphi = \sinh x$, $\cosh x$.

The Jacobi equation in the combustion case ($\alpha > 0$) has the form

$$\frac{d^2\tau}{d\eta^2} + \frac{2}{\mathrm{ch}^2\,\eta}\,\tau = 0, \tag{11}$$

where η is a dimensionless coordinate ($\eta_- \leq \eta \leq \eta_+$) and $\tau = \alpha T'$ is a dimensionless temperature perturbation. The solution of (11) satisfying the conditions $\tau(\eta_-) = 0$ and $d\tau/d\eta(\eta_-) = 1$ is the following [6]:

$$\tau = -\,\mathrm{th}\,\eta_- \{1 + [(\eta_- - \mathrm{cth}\,\eta_-) - \eta]\,\mathrm{th}\,\eta\}. \tag{12}$$

For a nonmonotonic change in T across a layer when $\eta = 0$ there corresponds max T, $\eta_- \leq 0 \leq \eta_+$, and

$$-\eta_- + \eta_- = \alpha_-\,\mathrm{ch}\,\eta_- = \alpha_+\,\mathrm{ch}\,\eta_- > 0, \quad \alpha_-/\alpha_+ = \mathrm{ch}\,\eta_+/\mathrm{ch}\,\eta_- = \beta < 1,$$

where $\alpha_\pm$ and β are certain constants. We hence obtain

$$|\eta_-| = \alpha_-\,\mathrm{ch}\,|\eta_-| - \mathrm{Ar\,ch}\,(\beta\,\mathrm{ch}\,|\eta_-|). \tag{13}$$

Depending on the values of α_- and β, Eq. (13) in $|\eta_-|$ has either two or no solutions. The derivative of the right-hand side of (13) with respect to $|\eta_-|$ is

$$\psi \equiv \mathrm{th}\,|\eta_-|\,[|\eta_-| - (\mathrm{cth}\,\eta_+ - \eta_+)],$$

while the derivative of the left-hand side of (13) with respect to $|\eta_-|$ is one. Since $\psi \leq 1$ corresponds to a lesser and $\psi \geq 1$ to a greater solution of $|\eta_-|$ in the presence of solutions, it follows from (12) that $\tau(\eta_+) \geq 0$ holds for the first solution and $\tau(\eta_+) \leq 0$ for the second. It can be seen that the inequality $\tau(\eta) > 0$ is satisfied for $\eta_- < \eta < \eta_+$ for the first solution while the sign of τ changes in the same range of variation of η for the second solution. In conformity with the theory of sufficient conditions for the weak minimum of a functional [6], this means that $\delta^2 J_0 > 0$ for the first solution but the sign of $\delta^2 J_0$ is uncertain for the second. The same result is obtained for a monotonic change in T across the layer also.

Therefore, even in the combustion case when the sufficient condition for stability (7) is not satisfied, the mode being realized physically (the first solution) corresponds to absolute stability of the system relative to the temperature perturbations under consideration.

The evolution criterion used here $\partial J/\partial t < 0$ has no relation to the known principle of the minimum of entropy occurrence (see [8, 9], for example), and is a result of the parabolicity of the nonstationary equation (4) in the long run. The validity of this criterion is not constrained by the condition of constancy of the heat-conduction coefficient $\varkappa$. However, for $\varkappa$ = const the criterion is true even taking into account a convective term of the form $\rho c v\, \partial T/\partial x$, which is not possible for the principle of the minimum of the rate of entropy occurrence [8, 9].

LITERATURE CITED

1. G. I. Barenblatt and Ya. B. Zel'dovich, Prikl. Mat. Mekh., 21, No. 6, 856 (1957).
2. A. G. Istratov and V. B. Librovich, Prikl. Mat. Mekh., 27, No. 2, 343 (1963).
3. F. G. Bass, V. S. Bochkov, and Yu. G. Gurevich, Zh. Eksp. Teor. Fiz., 58, No. 5, 1814 (1970).
4. Yu. Yu. Abramov and G. G. Gladush, Prikl. Mekh. Tekh. Fiz., No. 2, 24 (1974).
5. Yu. S. Levitan, Teplofiz. Vys. Temp., 15, No. 2, 261 (1977).
6. I. M. Gel'fand and S. V. Fomin, Calculus of Variations, Prentice-Hall (1963).
7. E. Kamke, Handbook of Ordinary Differential Equations [in German], Chelsea Publ.
8. S. R. De Groot and P. Masur, Non-Equilibrium Thermodynamics, American Elsevier (1962).
9. I. Gyarmati, Non-Equilibrium Thermodynamics, Springer-Verlag (1970).